7 Hermitian spaces and quaternions

Monday, September 28, 2020 5:03 PM

Last time we discussed in detail orthogonality, inner products, etc. in Euclidean spaces, and mentioned that something comparable holds in Hermitian spaces. We won't be covering the generalization in detail, but here are a few useful analogues.

$x \in \mathbb{R}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$a + bi = z \in \mathbb{C}$

$\bar{z} = a - bi$

$|z| = z\bar{z} = a^2 + b^2.$

Linear $f(u+v) = f(u) + f(v)$, $f(\lambda u) = \lambda f(u)$

Semilinear (Def 13.1) $f(u+v) = f(u) + f(v)$, $f(\lambda u) = \bar{\lambda} f(u)$.

Bilinear form (Def 11.1)

$\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$

$\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$

$\varphi(\lambda u, v) = \lambda \varphi(u, v)$

$\varphi(u, \lambda v) = \lambda \varphi(u, v)$

Sesquilinear form (Def. 13.2)

$\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$

$\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$

$\varphi(\lambda u, v) = \lambda \varphi(u, v)$

$\varphi(u, \lambda v) = \bar{\lambda} \varphi(u, v)$

(linear in first argument, semilinear in 2nd)

Symmetric $\varphi(u, v) = \varphi(v, u)$

Hermitian Sesquilinear and $\varphi(v, u) = \overline{\varphi(u, v)}$.

Note: A sesquilinear form is $\mathbb{R}$-bilinear

Def. 13.4 Given a complex vector space E, a Hermitian form $\varphi : E \times E \to \mathbb{C}$ is positive if $\varphi(u, u) \geq 0$ $\forall u \in E$ and positive definite if $\varphi(u, u) > 0$ for all $u \neq 0$. A pair $\langle E, \varphi \rangle$ is called a Hermitian (or unitary) space if φ is pos def.

Def. 13.5 The matrix $G = (g_{ij})$ where $g_{ij} = \varphi(e_i, e_j)$ over a basis $(e_1, \ldots, e_n)$ of E is called the Gram matrix of the Hermitian product φ w.r.t. $(e_1, \ldots, e_n)$.

Rem. 11: A Hermitian matrix G is one s.t. $G = G^* = \overline{(G^T)}$.

Recall: A Hermitian matrix G is one s.t. $G = G^* = \overline{(G^T)}$.

Note: A Hermitian pos. def. matrix A defines a Hermitian form $\langle x, y \rangle = y^* A x$ which is pos. def.

We get nearly all of the same results for Hermitian spaces that we did with Euclidean spaces, i.e. duality, orthogonality, etc.

Thm 13.1/13.6 Let E be a Hermitian space E. The map $\flat : E \to E^*$ defined s.t. $\flat(u) = \varphi_u^l = \varphi_u^r$, where $\varphi_u^l(v) = \overline{u \cdot v}$ and $\varphi_u^r(v) = v \cdot u$ is semilinear and injective. When E is also of finite dim., $\flat : \overline{E} \to E^*$ is a canonical isomorphism.

($\overline{E}$ is the space with the same set as E and the same addition operation, but where multiplication $(\lambda, u) \mapsto \bar{\lambda} u$.

Prop. 13.6/13.7 If E is a Hermitian space of finite dim, the every linear form $f \in E^*$ corresponds to a unique $v \in E$ s.t.

$$f(u) = u \cdot v, \quad \forall u \in E.$$

If f is not the 0 form, then $\operatorname{Ker} f$ = hyperplane $H = \{$ set of vectors orthogonal to $v \}$.

Def. 13.6 Given a Hermitian space E of finite dim., for every linear map $f: E \to E$, the unique linear map $f^* : E \to E$ s.t.

$$f^*(u) \cdot v = u \cdot f(v), \quad \forall u, v \in E$$

is called the adjoint of f. w.r.t. the Hermitian inner prod.

Equiv. to $f(u) \cdot v = u \cdot f^*(v)$.

Note: $f^{**} = f$

$(f+g)^* = f^* + g^*$

If $f = f^*$, then we call f self-adjoint.

$$(f+g)^* = f^* + g^*$$ call f self-adjoint.

$$(\lambda f)^* = \bar{\lambda} f^*$$

$$(g \circ f)^* = f^* \circ g^*.$$

Def. 13.7 $f: E \to F$ is a unitary transformation (or linear isometry) if it is linear and $\|f(u)\| = \|u\|$ $\forall u \in E$, where $\|u\| = \sqrt{\varphi(u,u)}$.

Prop. 13.13/13.14 Given $\forall f: E \to F$, where E and F are Hermitian spaces of the same finite dim, then the following are equiv.

(1) If f is a linear map, and $\|f(u)\| = \|u\|$ $\forall u \in E$.

(2) $\|f(v) - f(u)\| = \|v-u\|$ and $f(iu) = if(u)$ $\forall u, v \in E$ (viz $f(0)=0$ in Euclidean space)

(3) $f(u) \cdot f(v) = u \cdot v$ $\forall u, v \in E$.

Define: $A^* = \overline{(A^T)} = (\bar{A})^T$, the adjoint of a matrix.
(i.e. conjugate of transpose, or transpose of conjugate)

Def. 13.9: A complex $n \times n$ matrix is a unitary matrix if
$$AA^* = A^*A = I_n.$$

Equiv.: $A^{-1} = A^*$.

Def. 13.11: Given any complex $n \times n$ matrix A, a QR-decomposition is any pair of $n \times n$ matrices (U, R) where U is a unitary matrix and R is an upper triangular matrix s.t. $A = UR$.

We can use the exact same Gram-Schmidt orthonormalization proof for invertible matrices.

But, reflections are a lot trickier in Hermitian spaces.

Def. 13.12: Let E be a Hermitian space of finite dim. For any hyperplane H, for any $w \neq 0$ orthogonal to H, so that $E = H \oplus G$, where $G = \mathbb{C}w$, a Hermitian reflection about H of angle θ is a linear map of the form

$$\rho_{H,\theta}(u) = p_H(u) + e^{i\theta} p_G(u),$$

for any unit complex number $e^{i\theta} \neq 1$. For any $0 \neq w \in E$, we denote by $\rho_{w,\theta}$, the Hermitian reflection given by $\rho_{H,\theta}$, where H is the hyperplane orthogonal to w.

You can use Hermitian reflections to get QR decompositions as well, though it's rather more involved.

Def. 13.10 Given a Hermitian space E of dim n, the set of isometries $f: E \to E$ forms a subgroup of $GL(E, \mathbb{C})$ denoted by $U(E)$, or $U(n)$ when $E = \mathbb{C}^n$, called the unitary group of E. For every isometry, $|\det(f)| = 1$. If $\det(f) = 1$, we call that a rotation, or proper isometry, or proper unitary transformation, and those form another subgroup of the special linear group $SL(E, \mathbb{C})$ (and of $U(E)$), which we denote $SU(E)$ or $SU(n)$, if $E = \mathbb{C}^n$., the special unitary group of E.

Viz. $O(n)$, the orthogonal group of $\mathbb{R}^n$
$SO(n)$, the special orthogonal group of $\mathbb{R}^n$. (rotations in $\mathbb{R}^3$)

One particular special case is the relationship between unit quaternions and rotations, which is used often in practice in computer graphics.

Def 15.1 The unit quaternions are the elements of the group SU(2):

$$SU(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \text{ s.t. } \alpha, \beta \in \mathbb{C}, \ \alpha\overline{\alpha} + \beta\overline{\beta} = 1 \right\}.$$

$$\left(\begin{bmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^* = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} = \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \right.$$
$$\left. \Rightarrow \delta = \overline{\alpha}, \ \overline{\gamma} = -\beta \Rightarrow \gamma = -\overline{\beta}. \right)$$

The quaternions are the elements of the real vector space $\mathbb{H} = \mathbb{R}\, SU(2)$.

Let $\underset{\sim}{1}, \underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}$ be the matrices

$$\underset{\sim}{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underset{\sim}{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \underset{\sim}{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \underset{\sim}{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then $\mathbb{H} = \{ X = a\underset{\sim}{1} + b\underset{\sim}{i} + c\underset{\sim}{j} + d\underset{\sim}{k}, \quad a, b, c, d \in \mathbb{R} \}$.

Notation: While working in the quaternions, let's drop the [~] below $1, i, j, k$

Can verify
$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = -ji = k$$
$$jk = -kj = i$$
$$ki = -ik = j.$$

Quaternions are a generalization of the complex numbers and turn out to be a skew field (non commutative field).

$\mathbb{H}$ is isomorphic to $\mathbb{R}^4$ as a vector space.

Def. 15.2 A concise notation for the quaternion $X = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$ is defined by $\alpha = a + ib$ and $\beta = c + id$ is

$$X = [a, (b, c, d)], \text{ where}$$

a is the scalar part of X and (b, c, d) is the vector part of X.

Note that $X^* = [a, -(b,c,d)]$, which we will also denote $\bar{X}$, the conjugate of X.

If X is a unit quaternion, then $\bar{X}$ is the multiplicative inverse of X.

(Lie algebra satisfying Lie bracket)

Def. 15.3 The real vector space $\mathfrak{su}(2)$ of 2×2 skew Hermitian matrices with 0 trace is given by

$A^* = -A$

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix} \middle| (x,y,z) \in \mathbb{R}^3 \right\}$$

Def. 15.4 The adjoint representation of the group $SU(2)$ is the group homomorphism $Ad: SU(2) \to GL(\mathfrak{su}(2))$ defined s.t. $\forall q \in SU(2)$, with $q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, we have $Ad_q(A) = qAq^*$, $A \in \mathfrak{su}(2)$

where q^* is the inverse of q, $q^* = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$, $qq^* = 1$.

Need to verify $Ad_q : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is invertible and Ad is a group homomorphism.

Then we can embed $\mathbb{R}^3$ into $\mathbb{H}$ by (we care about only "pure" quaternions)

$$\Psi(x,y,z) = \begin{pmatrix} ix & y+iz \\ -y+iz & -ix \end{pmatrix}.$$

Then q defines the map $\rho_q : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\rho_q(x,y,z) = \Psi^{-1}(q\Psi(x,y,z)q^*).$$

It turns out this ρ_q is a rotation, and we can represent rotations in $SO(3)$ by the adjoint representation of $SU(2)$.